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2006 J. Phys.: Condens. Matter 18 2655

(http://iopscience.iop.org/0953-8984/18/9/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **18** (2006) 2655–2663 doi:10.1088/0953-8984/18/9/005

# Localization and the effective mass of a vortex

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Received 30 August 2005
Published 17 February 2006
Online at stacks.iop.org/JPhysCM/18/2655

**Abstract**
In the problem of a vortex escaping from a metastable potential, we derive the localization criterion of a vortex at finite dissipation and temperature by analysing both the crossover temperature formula and the escape rate formula. In the absence of a pinning potential in the stable direction, this criterion shows that a vortex will be localized in a metastable potential when the Magnus force is strong enough. Using the concept of this localization criterion, the effective mass of a vortex can be defined and interpreted. Moreover, the role of pinning and dissipation in the process of vortex escape can also be discussed.

## 1. Introduction

A quantized vortex is a topological object that generally exists in a variety of materials such as a Bose superfluid $^4$He [1, 2], type-II superconductors [3] and a Fermi superfluid $^3$He [4]. The dynamical properties of vortices are widely accessible to experimental studies in both classical and quantum regimes and their importance has long been realized [5–8]. In high-temperature superconductors, the physics of vortices also shows many new aspects not encountered in conventional superconductors [9]. One of the main problems in type-II superconductors is that when the magnetic field is applied the quantized magnetic flux, which is always associated with a vortex line, penetrates the superconductor and therefore a vortex movement (due to the presence of the Lorentz-like Magnus force [10–12]) causes an unfavourable condition of resistance in the superconductor. To prevent this resistance, the vortex needs to be pinned. The greater the pinning force, the greater the level of current that can flow in the zero-resistance state. Another big problem concerns the topic of the intrinsic mass or the inertial mass of a vortex [13–15]. This intrinsic mass has been identified as corresponding to different origins such as core mass [16], the electromagnetic field mass [16–20] and the strain field inertial

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mass [21]. Moreover, the theoretical estimates of the mass range from zero [22], to finite [23], to infinite [17, 18].

In this paper we use the model Hamiltonian proposed by Ao and Thouless [24], which contains the assumed non-zero finite vortex mass, to make an analytical study of a vortex escaping from a metastable potential in the presence of the Magnus force, pinning and dissipation. This study leads to many conclusions concerning the roles of pinning and dissipation and especially the localization and the effective mass of a vortex. We also show that although the non-zero mass is used here, the resulting effective mass expression will be shown to be the same as the one that uses the assumed zero mass by setting this non-zero mass to zero at the end of our analysis. Now, let us start with the Hamiltonian for a vortex moving in a two-dimensional $x$–$y$ plane, which can be regarded as a point particle, in the form [24]

$$H = \frac{1}{2M}\left|\vec{P} - q_v\vec{A}(\vec{r})\right|^2 + V(\vec{r}) + \sum_j \left(\frac{1}{2m_j}\left|\vec{P}_j\right|^2 + \frac{1}{2}m_j\omega_j^2\left|\vec{q}_j - \frac{c_j}{m_j\omega_j^2}\vec{r}\right|^2\right), \tag{1}$$

with the vector potential $\vec{A}$ determined by $\vec{\nabla} \times \vec{A} = (M\Omega/q_v)\hat{z}$. Here $q_v = +1\ (-1)$ stands for the vorticity parallel (antiparallel) to the unit vector $\hat{z}$ in the $z$ direction, $M$ is the intrinsic vortex mass and $\Omega$ is the frequency dimensional parameter which is equal to $q_v h\rho_s d/2M$ for a vortex in a superconductor (where $\rho_s$ is the superfluid electron number density) or $q_v h\rho_s d/M$ for a vortex in a superfluid (where $\rho_s$ is the superfluid atom number density). Here $h$ and $d$ are the Planck constant and the thickness of the sample (e.g. the thickness of the superconductor film), respectively. Note that since vortex motion under the Magnus force is similar to the motion of an electron in the presence of a magnetic field, the results obtained in this paper can be directly used in the problem of an electron escaping provided that $\Omega = eB/M$ and $q_v$ is replaced by an electron charge $e$.

Equation (1) can be explained as follows. The vector potential $\vec{A}$ reflects the existence of the vortex velocity-dependent part (VVDP) of the Magnus force $\vec{F}_M = M\Omega(\vec{v}_s - \dot{\vec{r}}) \times \hat{z}$ which depends on the relative velocity between the superfluid velocity $\vec{v}_s$ (which is assumed, without loss of generality, to be parallel to the $x$-axis) and the vortex velocity $\dot{\vec{r}}$. The Magnus force can be derived by various methods, e.g. by Thouless and co-workers [10], Wexler [11] and Sa-yakanit [12]. It is clear that the frequency dimensional parameter $\Omega$ we have just defined represents the strength of the Magnus force (or the Lorentz force for the problem of an electron escaping). The superfluid velocity-dependent part (SFVDP) of the Magnus force will contribute to the vortex potential $V(\vec{r})$. By following [24], we shall put the vortex potential $V(\vec{r})$, which contains both the contribution from the SFVDP Magnus force and the pinning centres, in the form

$$V(\vec{r}) = V(y) + \tfrac{1}{2}k_x x^2. \tag{2}$$

The pinning potential in the $x$ direction is approximated by a harmonic potential characterized by the parameter $k_x$. In this paper, the potential $V(y)$ consisting of the contributions from the SFVDP Magnus force and the pinning potential in the $y$ direction is assumed to be of the *metastable cubic-plus-quadratic* form with a metastable point at $y = 0$ [25, 26]. This metastable potential is characterized by two parameters: (i) $\omega_0$, the frequency of the small oscillation about $y = 0$ of the potential $V(y)$, and (ii) $\omega_b$, the frequency of the small oscillation about $y = y_b$ ($V(y_b)$ is equal to the potential at the barrier top) of the inverted potential $-V(y)$. Note that in the problem of an electron escaping the potential $V(y)$ consists of the pinning potential in the $y$ direction only because an electron can feel the Lorentz force by its own velocity. The last term in equation (1) represents the dissipative environment of a vortex consisting of a set of harmonic oscillators as formulated in [27]. The effect of the dissipative environment is specified by the following spectral function:

$$J(\omega) = \pi \sum_j \frac{c_j^2}{2m_j\omega_j}\delta(\omega - \omega_j). \tag{3}$$

In our problem of escaping, the Euclidean action corresponding to the Hamiltonian (1) is independent of the choice of gauge since the boundary condition $\vec{r}(0) = \vec{r}(\beta\hbar)$, where $\beta = 1/k_B T$ is the inverse temperature, is required. For this reason, we can choose any form of vector potential whenever it satisfies the relation $\vec{\nabla} \times \vec{A} = (M\Omega/q_v)\hat{z}$. As in [24], the vector potential will be chosen in the form $\vec{A} = (M\Omega/q_v)(y, 0, 0)$. The Euclidean action corresponding to the Hamiltonian (1) with this form of the vector potential is [24]

$$S^{\mathrm{E}} = \int_0^{\beta\hbar} \mathrm{d}\tau \left[ \tfrac{1}{2}M\left|\dot{\vec{r}}\right|^2 + \mathrm{i}M\Omega\dot{x}y + V(y) + \tfrac{1}{2}k_x x^2 + \sum_j \left( \frac{1}{2}m_j \left|\dot{\vec{q}}_j\right|^2 + \frac{1}{2}m_j\omega_j^2 \left|\vec{q}_j - \frac{c_j}{m_j\omega_j^2}\vec{r}\right|^2 \right) \right]. \tag{4}$$

After integrating the environmental and $x$ degrees of freedom of a vortex, the reduced thermodynamic description in the metastable direction, i.e. the $y$ direction, can be known via the reduced partition function

$$Z_{\mathrm{d}}(\beta\hbar) = \oint \mathrm{D}[y(\tau)] \exp(-S_{\mathrm{eff}}^{\mathrm{E}}[y(\tau)]/\hbar), \tag{5}$$

where

$$S_{\mathrm{eff}}^{\mathrm{E}}[y(\tau)] = \int_0^{\beta\hbar} \left[\tfrac{1}{2}M\dot{y}^2 + V(y)\right] \mathrm{d}\tau + \tfrac{1}{2}\int_0^{\beta\hbar}\int_0^{\beta\hbar} \left[K(|\tau - \tau'|) + g(\tau - \tau')\right] \times \left[y(\tau) - y(\tau')\right]^2 \mathrm{d}\tau' \,\mathrm{d}\tau. \tag{6}$$

From [24], $K(\tau)$ and $g(\tau)$ are called the normal and anomalous damping kernel, respectively. They are expressed as

$$K(\tau) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\omega\, J(\omega) \frac{\cosh\left[\omega(\beta\hbar/2 - \tau)\right]}{\sinh\left[\omega\beta\hbar/2\right]}, \tag{7}$$

and

$$g(\tau) = \frac{M\Omega^2}{2\beta\hbar}\sum_{n=-\infty}^{\infty}\left(\frac{M\omega_x^2 + \xi_n}{M\nu_n^2 + M\omega_x^2 + \xi_n}\right)\mathrm{e}^{\mathrm{i}\nu_n\tau}. \tag{8}$$

Here

$$\xi_n = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\, \frac{J(\omega)}{\omega}\frac{2\nu_n^2}{\omega^2 + \nu_n^2}; \qquad \nu_n = 2n\pi/\beta\hbar, \quad \omega_x^2 = k_x/M. \tag{9}$$

In the problem of escaping, one of the important quantities is the *escape rate* (denoted by $K$). According to Affleck [28], the escape rate formula is divided into two forms separated by the crossover temperature (denoted by $T_0$) as follows:

$$K = -\frac{2}{\hbar}\,\mathrm{Im}\,F; \qquad \text{for } T < T_0, \tag{10}$$

and

$$K = -\frac{2}{\hbar}\frac{\beta}{\beta_0}\,\mathrm{Im}\,F; \qquad \text{for } T > T_0, \tag{11}$$

where $\beta_0 = 1/k_B T_0$ is the inverse crossover temperature and $F$ is the free energy which is related to the reduced partition function (5) by $Z_d = \exp(-\beta F)$. In equations (10) and (11), the imaginary part of the free energy ($\mathrm{Im}\, F = -(1/\beta)\,\mathrm{Im}(\ln Z_d)$) can be calculated by using the analytic continuation method pioneered by Langer [29]. The crossover temperature, which is mainly used for the analysis in this paper, is the temperature where the dominant mechanism of the escape process changes from thermal activation to quantum tunnelling. In the functional integral point of view [25, 26], the crossover temperature is the temperature where the dominant trajectory of the functional integral for $Z_d$ changes from the trivial trajectory ($y = 0$ and $y = y_b$) to the bounce trajectory (the back and forth trajectory in the inverted potential). In other words, if we decrease the temperature from $T > T_0$ to $T < T_0$, then the corresponding dominant trajectory will be changed from trivial to bounce and the corresponding dominant physical mechanism of the escape process will roughly change from thermal activation to quantum tunnelling.

## 2. Existence of the crossover temperature

It is worthwhile now finding the equation for determining the crossover temperature $T_0$. The procedures are as follows. First, from the definition of the crossover temperature discussed above, it is clear that slightly below $T_0$ the bounce trajectory will be replaced by the harmonic oscillator which is the small oscillation about $y_b$ with the frequency $\omega_R = 2\pi/\beta_0\hbar$. Second, by using the variational principle $\delta S^E_{\mathrm{eff}} = 0$ where $S^E_{\mathrm{eff}}$ has already been defined in equation (6), the equation of motion is obtained. Third, by substituting the harmonic oscillator solution from the first step into the equation of motion in the second step, then with the help of equations (7) and (8), one can linearize the equation of motion and get

$$\omega_R^2 + \omega_R \hat{\gamma}_M(\omega_R) = \omega_b^2, \tag{12}$$

where $\omega_R = 2\pi/\beta_0\hbar = 2\pi k_B T_0/\hbar$, $\omega_b^2 = -V''(y_b)/M$, and

$$\hat{\gamma}_M(x) = \hat{\gamma}(x) + \frac{\Omega^2 x}{x^2 + \omega_x^2 + x\hat{\gamma}(x)}, \qquad \text{for all } x \geqslant 0, \tag{13}$$

where $\hat{\gamma}(x) = (2x/M\pi)\int_0^\infty [J(\omega)/\omega(\omega^2 + x^2)]\,\mathrm{d}\omega$ is the Laplace transform of the retarded friction [25, 26] (when the environment is represented by the Caldeira–Leggett model [27]). Note that $\hat{\gamma}_M(x)$, where the subscript M denotes the Magnus force, reduces to $\hat{\gamma}(x)$ when the Magnus force is absent (i.e. $\hat{\gamma}_M \to \hat{\gamma}$ as $\Omega \to 0$) and then equation (12) also reduces to $\omega_R^2 + \omega_R\hat{\gamma}(\omega_R) = \omega_b^2$ [25, 26], which is the equation for determining $T_0$ in the well-known case of a one-dimensional system. This should not be a surprise, since when $\Omega \to 0$ the motion in the unstable direction ($y$ direction) will be decoupled from the stable one ($x$ direction).

The interesting question is that how much we can investigate the behaviour of $T_0$ or equivalently $\omega_R$. Evidently, to know all of its behaviours one must find the roots of equation (12). Unfortunately, it may be impossible to find them since $\hat{\gamma}(x)$ contains an integral of $J(\omega)$ which has no specific form in general cases. However, just two of its properties are sufficient for us to investigate the important physical situations which will be described in the next two sections. The first property is the *uniqueness* of $T_0$. One can guess that if $T_0$ exists, then it should be unique since the crossover temperature is the temperature where the change in the dominant mechanism of the escape process roughly occurs and once this change has occurred it should not occur again. The second property is the *existence* of $T_0$. One may guess at first sight that $T_0$ always exists because when we decrease the temperature from high to very low, then the dominant mechanism of escape should roughly changed from thermal activation to quantum tunnelling at some temperatures. In order to check whether these guesses are correct or not, these two properties can be proved mathematically by looking at equation (12) carefully as follows. We rewrite equation (12) as $x\hat{\gamma}_{\mathrm{M}}(x) = \omega_b^2 - x^2$ ($\omega_{\mathrm{R}}$ is the root of this equation). Here, $x$ is confined only in the positive range i.e. $x \geqslant 0$ since $T_0$ is the absolute temperature which is always greater than or equal to zero. Notice that $\omega_b^2 - x^2$ is the continuous decreasing function on the positive range of $x$ and has a maximum value equal to $\omega_b^2$ at $x = 0$. Furthermore, one can prove from equation (13) through differentiating $x\hat{\gamma}_{\mathrm{M}}(x)$ that $x\hat{\gamma}_{\mathrm{M}}(x)$ is a continuous increasing function on a positive range of $x$. By these properties of $\omega_b^2 - x^2$ and $x\hat{\gamma}_{\mathrm{M}}(x)$, it is clear that the root of the equation $x\hat{\gamma}_{\mathrm{M}}(x) = \omega_b^2 - x^2$ exists and is unique if and only if

$$\lim_{x\to 0} x\hat{\gamma}_{\mathrm{M}}(x) \leqslant \omega_b^2. \tag{14}$$

Now, since the two properties of $T_0$ mentioned earlier have already been proved, it can then be summarized that *the crossover temperature exists and is unique if and only if the condition* (14) *is fulfilled*.

## 3. Vortex escaping formulae and the localization criterion

At this point, the following study of a vortex escaping is divided into two cases.

*First* ($\omega_x \neq 0$). In this case, one can prove from equation (13) that $\lim_{x\to 0} x\hat{\gamma}_{\mathrm{M}}(x) = 0 < \omega_b^2$, which is implied by condition (14) that *the unique crossover temperature $T_0$ always exists*. The existence of $T_0$ tells us that (i) there is a temperature where the dominant mechanism of escape will be roughly changed and (ii) the tunnelling rate (the escape rate when the dominant mechanism is quantum tunnelling) is always non-zero because of the existence of bounce trajectory. In this case, the escape rate $K$ for $T > T_0$ can be derived analytically by using the same methods as in [25, 26] as follows. First, replacing $y$ by its Fourier series, i.e. $y(\tau) = \sum_{n=-\infty}^{\infty} y_n \mathrm{e}^{\mathrm{i}\nu_n\tau}$, and substituting it into equation (6) and developing $V(y)$ in a Taylor series around $y = 0$ and $y = y_b$, the semiclassical effective action about $y = 0$ (denoted by $S_{\mathrm{eff}}^{\mathrm{E}(0)}[y]$) and $y = y_b$ (denoted by $S_{\mathrm{eff}}^{\mathrm{E}(b)}[y]$) can be expressed in the form

$$S_{\mathrm{eff}}^{\mathrm{E}(0)}[y] = \frac{M\beta\hbar}{2}\lambda_0^{(0)} y_0^2 + M\beta\hbar\sum_{n=1}^{\infty}\lambda_n^{(0)}|y_n|^2; \qquad \lambda_n^{(0)} = \nu_n^2 + \omega_0^2 + \nu_n\hat{\gamma}_{\mathrm{M}}(\nu_n), \tag{15}$$

and

$$S_{\mathrm{eff}}^{\mathrm{E}(b)}[y] = V_b\beta\hbar + \frac{M\beta\hbar}{2}\lambda_0^{(b)} y_0^2 + M\beta\hbar\sum_{n=1}^{\infty}\lambda_n^{(b)}|y_n|^2; \qquad \lambda_n^{(b)} = \nu_n^2 - \omega_b^2 + \nu_n\hat{\gamma}_{\mathrm{M}}(\nu_n), \tag{16}$$

where $V_b = V(y_b)$. Second, splitting the reduced partition function (5) into the contributions arising from the Gaussian fluctuations about the trivial paths $y = 0$ and $y = y_b$ and writing $Z_{\mathrm{d}} = Z_{\mathrm{d}}^{(0)} + Z_{\mathrm{d}}^{(b)}$, where $Z_{\mathrm{d}}^{(0)}$ and $Z_{\mathrm{d}}^{(b)}$ are the reduced partition functions which have the corresponding effective actions (15) and (16), respectively, the negative value of $\lambda_0^{(b)} = -\omega_b^2$ (after using the normalized functional measure in Fourier space [25] and Langer's thermodynamic method [29]) leads to an imaginary part of the free energy in the form

$$\mathrm{Im}\, F = -\frac{\omega_0}{2\beta\omega_b}\left(\prod_{n=1}^{\infty}\frac{\lambda_n^{(0)}}{\lambda_n^{(b)}}\right)\mathrm{e}^{-\beta V_b}. \tag{17}$$

Third, substituting equation (17) into (11), we finally obtain

$$K = \frac{\omega_0}{2\pi}\rho C_{\mathrm{qm}}\mathrm{e}^{-\beta V_b}, \tag{18}$$

where $\rho = \omega_R/\omega_b$ and $C_{qm} = \prod_{n=1}^{\infty} \lambda_n^{(0)}/\lambda_n^{(b)} \geqslant 1$ is called the quantum correction factor or the quantum-mechanical enhancement factor because it describes the quantum effects (i.e. tunnelling and increasing the average energy in the well) which enhance the escape rate.

Noticing from equation (13) that $\hat{\gamma}_M$ increases as the Magnus force strength (characterized by $\Omega$) increases. For this reason, one can conclude from equation (12) that $\omega_R$ or $\rho$ decreases when the Magnus force strength increases and, by the definition of $C_{qm}$ itself, $C_{qm}$ also decreases when the Magnus force strength increases. These imply that *the VVDP Magnus force tends to decrease the escape rate*. In contrast, *the pinning potential in the x direction tends to increase the escape rate* since, from equation (13), $\hat{\gamma}_M$ decreases as $\omega_x$ increases. Although these conclusions can be used when $T > T_0$ (because $K$ in equation (18) is valid for $T > T_0$ only), it may be used when $T < T_0$ too. This stems from the fact that since the correction factor $C_{qm}$ describes the quantum effect on the escape process including quantum tunnelling, the effects of the Magnus force and the pinning potential in the $x$ direction on tunnelling rate should be the same as on $C_{qm}$. As described above, $C_{qm}$ decreases (increases) when the Magnus force strength (pinning potential in the $x$ direction) increases. These imply (as in the case of $T > T_0$) that *the VVDP Magnus force tends to decrease the tunnelling rate while the pinning potential in the x direction tends to increase the tunnelling rate*. Moreover, *both pinning and dissipation tend to suppress the influence of the VVDP Magnus force on vortex escaping* since $\Omega$ is in the numerator while $\omega_x$ and $\hat{\gamma}$ (which contains an integral of $J(\omega)$) are in the denominator of the second term of equation (13).

*Second* ($\omega_x = 0$). In this case, one can prove from equation (13) that $\lim_{x\to 0} x\hat{\gamma}_M(x) = \Omega^2/[1 + (2/M\pi)\int_0^{\infty}(J(\omega)/\omega^3)\,d\omega]$. So, from condition (14), it is clear that the crossover temperature does not exist if

$$\frac{\Omega^2}{1 + \frac{2}{M\pi}\int_0^{\infty}\frac{J(\omega)}{\omega^3}\,d\omega} > \omega_b^2. \tag{19}$$

Condition (19) tells us that *the crossover temperature does not exist if the Magnus force strength is large enough*. This non-existence of a crossover temperature implies that the bounce trajectory does not exist and, hence, the tunnelling rate must vanish. Now, an interesting question arises: although the tunnelling rate vanishes, is it possible that the escape process, when condition (19) is fulfilled, will be dominated by thermal activation over the entire range of temperature? The answer is no, because of the fact that the value of $\lambda_0^{(b)}$ is now equal to $-\omega_b^2 + \Omega^2/[1 + (2/M\pi)\int_0^{\infty}(J(\omega)/\omega^3)\,d\omega]$ which is greater than zero when condition (19) is fulfilled. This positive value of $\lambda_0^{(b)}$ makes the free energy finite and real which implies that the escape rate must vanish or, equivalently, a vortex must be localized in the well. The above discussion can be summarized that *if* $\omega_x = 0$ *and the condition* (19) *is fulfilled, then the vortex must be localized in the well*. For this reason we shall call condition (19) the *localization criterion*, and since the derivation of this criterion is done irrespective of temperature and dissipation, it must be valid at any finite dissipation and temperature. By using $M\omega_b^2 = -V''(y_b)$, the localization criterion (19) can be written in the form

$$\frac{(M\Omega)^2}{M + \frac{2}{\pi}\int_0^{\infty}\frac{J(\omega)}{\omega^3}\,d\omega} > \left|V''(y_b)\right|, \tag{20}$$

where, by the definition of $\Omega$, $M\Omega$ is the mass-independent parameter, e.g. $M\Omega = q_v h\rho_s d/2$ for a vortex in a superconductor. Note that, for a vortex in a superconductor, the localization criterion (20) reduces to the localization criterion in the case of no pinning at zero temperature given by Ao and Thouless [24] when the dissipation is absent, i.e. $J(\omega) = 0$. In the case when the criterion (19) is violated, the escape rate, both for $T > T_0$ and $T < T_0$, does not vanish. The escape rate for $T > T_0$ in this case (denoted by $\tilde{K}$) can be derived by using the same methods in the first case as

$$\tilde{K} = \frac{\omega_0}{2\pi}\tilde{\rho}C_{\mathrm{qm}}\mathrm{e}^{-\beta V_b}; \qquad \tilde{\rho} = \omega_{0\mathrm{M}}\omega_{\mathrm{R}}/\omega_0\omega_{b\mathrm{M}}, \tag{21}$$

where

$$\omega_{0\mathrm{M}}^2 = \omega_0^2 + \Omega^2 \bigg/ \left[1 + (2/M\pi)\int_0^\infty (J(\omega)/\omega^3)\,\mathrm{d}\omega\right],$$

and

$$\omega_{b\mathrm{M}}^2 = \omega_b^2 - \Omega^2 \bigg/ \left[1 + (2/M\pi)\int_0^\infty (J(\omega)/\omega^3)\,\mathrm{d}\omega\right] > 0.$$

The subscript M on $\omega_{0\mathrm{M}}$ and $\omega_{b\mathrm{M}}$ denotes the abbreviated name of Magnus force due to its effect via the parameter $\Omega$. Note that $\omega_{0\mathrm{M}} \to \omega_0$ and $\omega_{b\mathrm{M}} \to \omega_b$ as $\Omega \to 0$ imply that $\tilde{K} \to K$ as $\Omega \to 0$. For $\Omega \to 0$, the localization criterion is always violated which implies that a vortex must escape from a metastable potential with a specific non-vanishing escape rate at any temperature. Note also that since $1 + (2/M\pi)\int_0^\infty (J(\omega)/\omega^3)\,\mathrm{d}\omega = M^*/M$ ($M^*$ is the effective mass defined in section 4) and $M\Omega$ is mass independent, $\tilde{K}$ is still well defined by equation (21) even when $M \to 0$. The situation where $M = 0$ is set will be discussed further in the next section and also in the conclusions.

The above two cases show that the pinning potential in the $x$ direction is an important quantity, because when $\omega_x \neq 0$ the escape rate is non-zero for any magnitude of the VVDP Magnus force while the escape rate for $\omega_x = 0$ is zero for a strong enough VVDP Magnus force. In other words, for $\omega_x = 0$, the VVDP Magnus force, which has sufficient strength, renormalizes the original metastable potential to the stable one. In the classical point of view, the pinning potential in the $x$ direction can bend the trajectory of a vortex in such a way that it helps a vortex to escape from the well while the VVDP Magnus force tends to trap a vortex in the well by keeping it in a circular motion.

## 4. Effective mass of a vortex

The effective mass of a vortex can be defined by using the localization criterion (20) as follows. Noticing that the $M\Omega$ term in the numerator of (20) is mass independent, the mass-dependent term is therefore only in the denominator. Hence, the *effective mass* (denoted by $M^*$) can be defined as

$$M^* := M + \frac{2}{\pi}\int_0^\infty \frac{J(\omega)}{\omega^3}\,\mathrm{d}\omega \tag{22}$$

so that the criterion (20) becomes

$$\frac{(M\Omega)^2}{M^*} > \left|V''(y_b)\right|. \tag{23}$$

From criterion (23), the effective mass can then be interpreted that since $M^* = M$ in the absence of dissipation (see equation (22)), a damped vortex (a vortex in contact with the environment) behaves as if it is an undamped vortex (a vortex which is free from the environment) of the new bigger mass called the effective mass when it decides to escape from the well. This effective mass is equal to the intrinsic mass plus the extra mass originating from the environment since it depends on the spectral function. Note that this extra mass, which is equal to the effective mass when the intrinsic mass $M$ vanishes, is equal to the effective vortex mass given by Han *et al* [30] (the spectral function defined here is equal to $\pi/2$ times the one defined in [30]). Complementing their derivation, we show that the anomalous damping kernel in the effective action (6), pertaining to the transverse motion of a vortex, *does not affect the effective mass* defined in equation (22) since it is independent of the Magnus force strength $\Omega$. Moreover, our derivation of the effective mass via the localization criterion does not require the *local in time* assumption which demands that a vortex must be moving sufficiently slowly or along a straight line. To understand more about the effective mass, we first consider our system in the new masses $\mu_j$ and new coordinates $\vec{\tilde{q}}_j$ for the environment [31] given by

$$\vec{\tilde{q}}_j = \frac{m_j \omega_j^2 \vec{q}_j}{c_j}, \qquad \mu_j = \frac{c_j^2}{m_j \omega_j^4}. \tag{24}$$

From equation (24), the Hamiltonian (1) can be rewritten as

$$H = \frac{1}{2M} \left| \vec{P} - q_v \vec{A}(\vec{r}) \right|^2 + V(\vec{r}) + \frac{1}{2} \sum_j \mu_j \left[ \left| \dot{\vec{\tilde{q}}}_j \right|^2 + \omega_j^2 \left| \vec{\tilde{q}}_j - \vec{r} \right|^2 \right]. \tag{25}$$

We can see that the model Hamiltonian (1), in fact, describes a vortex of mass $M$ with many masses $\mu_j$ attached by springs to its coordinate $\vec{r}$. From equations (3) and (24), the sum of $\mu_j$ can be written in the form

$$\sum_j \mu_j = \frac{2}{\pi} \int_0^\infty \frac{J(\omega)}{\omega^3}\, \mathrm{d}\omega. \tag{26}$$

It is clear, from equations (22) and (26), that the effective mass $M^*$ is equal to the total mass of the system which is composed of a vortex of intrinsic mass $M$ with many masses $\mu_j$. At this point, one may think that the coordinate of the effectively undamped vortex of mass $M^*$ may be the centre of mass coordinate which contains the total mass of the system. Although this conclusion may be possible, we cannot exactly do this since our definition of the effective mass does not come directly from the dynamical approach (it is defined via the localization criterion). However, some conclusions can be made for the case of sufficiently weak environmental coupling so that $\sum_j \mu_j \ll M$. In this case, the centre of mass coordinate of the system will approximately coincide with the original coordinate of a damped vortex at all times. For this reason, one can conclude in this case that the damped vortex of intrinsic mass $M$ can be effectively viewed as an undamped vortex of mass $M^*$ and the coordinate of this undamped vortex is identical to the centre of mass coordinate of the system which is approximately identical to the coordinate of an original damped vortex.

## 5. Conclusions

We have studied the influence of pinning, dissipation and the Magnus force on a vortex escaping through the localization criterion and the escape rate formulae and found (i) that at any temperature and dissipation, a vortex always escapes from the well when the pinning potential in the stable direction is present while it is localized in the well for strong enough VVDP Magnus force when the pinning potential in the stable direction is absent and (ii) at any temperature, the VVDP Magnus force tends to decrease the escape rate while the pinning potential in the stable direction tends to increase the escape rate. Moreover, both pinning and dissipation tend to suppress the influence of the VVDP Magnus force on vortex escaping. Also (iii) the effective mass of a vortex can be defined in the sense that when a damped vortex decides to escape from the well it can be effectively viewed as an undamped vortex of a new bigger mass called the effective mass. The effective mass is equal to the intrinsic mass plus the extra mass originating from the environment and can be viewed as the total mass of the system when considering the system in the appropriate coordinates and masses. For sufficiently weak environmental coupling, the whole system can be effectively viewed as one undamped vortex of effective mass described by the centre of mass coordinate which approximately coincides with the coordinate of an original damped vortex. It is interesting that when the intrinsic mass is taken as zero, our effective mass is equal to the one given by Han *et al* [30]. Besides the differences in methods and assumptions between their work and ours, another remarkable difference is that they claim that the intrinsic mass must be zero while our analysis requires a non-zero finite mass in the model Hamiltonian which can be set to be zero at the end of calculations. Hence, if a vortex really has a vanishing intrinsic mass in nature, the non-vanishing finite mass in the model Hamiltonian (1) may be interpreted as a *pseudo-mass* which can be set to be zero at some steps of the calculation in order to get the correct physical answers.

## Acknowledgment

The authors would like to thank the Thailand Research Fund for their financial support.

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